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Excess Noise in a Hopping Model for a Resistor with Quenched Disorder

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We consider a model for independent charged particles, hopping on a lattice with static disorder in the waiting times. The excess current noise is calculated and shown to be related to resistance noise and arising from mobility fluctuations. It is also related to the four point super-Burnett-function. The strength of the noise is calculated at small frequencies for weak disorder (classical long time tails) and for strong disorder, when it may behave like $1/f$. In that case the Hooge factor equals the fraction of deep trapping centers.

KEY WORDS: $1/f$ noise; mobility fluctuations; transport in disordered systems; hopping model; long time tails.

1. INTRODUCTION

When a steady current I flows through a resistor R two distinct sources of noise are observed.[1-4] The equilibrium current fluctuations (Johnson noise) have a white noise spectrum of magnitude $S_I(\omega) = 4kT/R$, where k is Boltzmann's constant and T the temperature. In addition there is an excess (nonequilibrium) current noise, in the presence of an electric field, with a power spectrum described by the phenomenological relation of Hooge,[2]

$$S_I^{\text{exc}}(\omega) = \alpha_H I^2/Nf \tag{1.1}$$

where $f = \omega/2\pi$ is the frequency and N is the number of charge carriers (charge e). The Hooge factor α_H is a dimensionless constant, typically of the order 10^{-4}–10^{-3} in metals and semiconductors. This $1/f$ scaling holds up to frequencies at which the excess noise is lost in Johnson noise and

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down to extremely low frequencies (e.g., 10^{-7} Hz [1,5]), limited by the time available for an experiment. The appearance of the factor $1/N$ in (1.1) supports the idea that $1/f$ noise is in general a bulk phenomenon. The proportionality $S_I^{\mathrm{exc}}(\omega) \sim I^2$ suggests that the current fluctuations result from (equilibrium) resistance fluctuations with a power spectrum[6]

$$S_R(\omega)/R^2 = S_I^{\mathrm{exc}}(\omega)/I^2 = \alpha_H/(Nf) \tag{1.2}$$

Since $S_R(\omega)$ is independent of the current, these (equilibrium) fluctuations are also present without an applied electric field. The Johnson noise voltage, passed through an appropriate band pass filter, is proportional to the resistance. Voss and Clarke[6] and Beck and Spruit[7] showed that resistance fluctuations can be measured by examining the fluctuations in the strength of the filtered Johnson noise power, and found $1/f$ noise as in (1.2).

Tremblay and Nelkin[8] put these ideas on a firmer theoretical footing and showed that such measurements probe the four-point current correlation in equilibrium.

Similar relationships between the nonequilibrium two-point current correlation function and the equilibrium four-point current correlation function (Burnett function) have been derived for a variety of models.[9-11]

The resistance can be expressed in the mobility μ or, equivalently, in the diffusion coefficient D, i.e., $R = (L^2/\mu N) = (L^2 kT/e^2 DN)$, where L is the length of the sample in the direction of the field. Fluctuations in R may occur through fluctuations in the number of carriers[12] or in the mobility.[2] There exist strong experimental indications that, in general, mobility fluctuations are responsible for $1/f$ noise in semiconductors[2] and (in combination with temperature fluctuations) also in metal films.[1]

The purpose of this paper is to explore to what extent the long time tails, as found in fluids and systems with static disorder (e.g., Lorentz gases)[13,14] could provide a possible mechanism for explaining the noise properties, described above. Thus we study the nonequilibrium current fluctuations and the equilibrium resistance fluctuations for a lattice random walk model with quenched disorder and show that they satisfy Eq. (1.2).

The basic idea, frequently used in the noise literature,[2,9,15] is that the excess noise originates at many independent sites, distributed homogeneously throughout the medium. Charge carriers are continuously entering and leaving the wells at these sites with a characteristic escape or waiting time $\tau_{\mathbf{n}}$ at the $\mathbf{n}$th lattice point. These $\tau_{\mathbf{n}}$ are considered to be independent random variables with a site independent waiting time distribution $\rho(\tau)$. The distribution of waiting times, especially of "deep" wells with long waiting times, determines the small-frequency behavior of the spectral density of current noise. The random distribution of these wells tries to model the spatial *fluctuations in the mobility* of the charge carriers, i.e., fluctuations in the local scattering throughout the medium, but does not model *fluctuations in the number of carriers*, as occurring in McWorter's model[12] or in the hopping models with conducting and trapping states at each lattice site.[16,9,11]

The study of hopping models with quenched disorder as a possible model for a resistor with excess current noise has been initiated by Lehr *et al.*[14] These authors have shown that hopping conduction on a weakly disordered chain leads to $1/\sqrt{f}$ noise. In the present paper their work is extended to higher-dimensional systems with weak and strong disorder. Preliminary results have already been reported in Ref. 17.

An interesting feature of some of the conduction models with quenched disorder (including d-dimensional Lorentz gases and several hopping models[13] and lattice gas models[10]) is that the long time tails in the current correlation function are of *dynamic origin*, and exist even if all moments $\langle \tau^n \rangle$ $(n = 0, 1, 2,...)$ of the waiting time distribution are finite (weak disorder).

A particular interesting example is the quantum mechanical Lorentz gas in d dimensions (a case of weak disorder), where the excess current noise behaves as $S_I^{\text{exc}}(\omega) \sim \omega^{(d-4)/2}$, as shown by Kirkpatrick and Dorfman.[18]

On the other hand, the long time tails in the continuous time waiting time models[19,16,9] are of *static origin*, i.e., they depend on the presence of algebraic tails (strong disorder) in the static waiting time distribution $\rho(\tau)$. If $\langle \tau^n \rangle$ $(n = 0, 1, 2,...)$ is finite, all tails are absent. A discussion of the dynamic versus static origin of long time tails is given in Ref. 20.

In the remaining part of this section we briefly review some concepts needed explicitly in the paper. The spectral densities or power spectra of interest here are the Fourier transforms of the current and resistance fluctuations. The instantaneous or fluctuating current $I_x(t)$ in the field direction can be identified as the spatially averaged velocity of N *independent* charge carriers

$$I_x(t) = (e/L) \sum_{j=1}^{N} v_{jx}(t) \tag{1.3}$$

If a (small) uniform electric field E is applied in the x direction, there exists a nonvanishing steady current $I = eN\langle v_x \rangle / L$, where $\langle \cdots \rangle$ is a steady state average. The spectral density of the longitudinal current fluctuations is defined as

$$S_I(\omega) = 4 \int_0^\infty dt \cos \omega t [\langle I_x(t) I_x(0) \rangle - \langle I_x \rangle^2] \tag{1.4}$$

where the current correlation for independent charge carriers is related to the nonequilibrium ($\varepsilon \neq 0$) velocity autocorrelation function (VACF) $\varphi_{\|}(t; \varepsilon)$ in the presence of a dimensionless field $\varepsilon = eEl/2kT$:

$$\langle I_x(t)\, I_x(0)\rangle - \langle I_x\rangle^2 = (e^2N/L^2)\, \varphi_{\|}(t; \varepsilon) \tag{1.5}$$

The longitudinal and transverse VACF's are defined as

$$\begin{aligned} \varphi_{\|}(t; \varepsilon) &= \langle v_x(t)\, v_x(0)\rangle - \langle v_x\rangle^2 \\ \varphi_{\perp}(t; \varepsilon) &= \langle v_\alpha(t)\, v_\alpha(0)\rangle \equiv \varphi_0(t) \qquad (\alpha \neq x) \end{aligned} \tag{1.6}$$

Thus the current noise is

$$S_I(\omega) = (4e^2N/L^2)\, \Re e\hat{\varphi}_{\|}(i\omega; \varepsilon) \tag{1.7}$$

where $\hat{f}(z) = \mathfrak{L}_z f(t)$ denotes the Laplace transform.

Next, we consider resistance fluctuations in absence of a field. Since the hopping model to be considered does not admit any fluctuations in N, we consider only fluctuations in the diffusion coefficient D. To define D fluctuations we introduce a fluctuating diffusion coefficient

$$D(t, \tau) = \tfrac{1}{2}\partial_t[x(t) - x(\tau)]^2 \tag{1.8}$$

close to a definition of Stanton and Nelkin.[9] Its expectation in a stationary initial ensemble,

$$\langle D(t, \tau)\rangle = \int_0^{t-\tau} dt' \langle v_x(t')\, v_x(0)\rangle \tag{1.9}$$

yields the diffusion coefficient D in the long time limit. The autocorrelation of D fluctuations in a stationary initial ensemble is then

$$\langle D(t, \tau)\, D(t', \tau')\rangle - \langle D(t, \tau)\rangle\langle D(t', \tau')\rangle \simeq C_D(t - t') \qquad (t, t' \text{ large}) \tag{1.10}$$

which involves a four-point correlation function. For *large values of t and t′* this function will only depend on the time difference $t - t'$. This limiting value will be denoted by $C_D(t - t')$. The spectral density of D fluctuations is therefore

$$S_D(\omega) = 4\Re e\hat{C}_D(i\omega) \tag{1.11}$$

Stanton and Nelkin[9] have in fact discussed the spectral density of the band-filtered Johnson noise power P, where $\langle P\rangle = (4kT/R)\, \Delta f$ is the average Johnson noise power, and Δf is the width of the frequency band. Under certain assumptions on the behavior of the four-point current correlation function or (equivalently for independent charge carriers) of the four-point velocity correlation function multiplied by N, these authors were able to show that

$$S_P(\omega)/\langle P\rangle^2 = S_D(\omega)/ND^2 \tag{1.12}$$

Here the correlation function of D fluctuations could be expressed as

$$C_D(t) = \int_0^\infty d\tau \int_0^\infty d\tau' \langle\langle v_x(t+\tau)\, v_x(t)\, v_x(\tau')\, v_x(0)\rangle\rangle \tag{1.13}$$

where $\langle\langle\cdots\rangle\rangle$ is the fourth cumulant, defined as

$$\langle\langle A_1A_2A_3A_4\rangle\rangle = \langle A_1A_2A_3A_4\rangle - \langle A_1A_2\rangle\langle A_3A_4\rangle - \langle A_1A_3\rangle\langle A_2A_4\rangle - \langle A_1A_4\rangle\langle A_2A_3\rangle \tag{1.14}$$

For the present model it will be verified (Section 5) that Stanton and Nelkin's assumptions regarding $\langle\langle\cdots\rangle\rangle$ in (1.13) are correct.

The outline of the paper is as follows. In Section 2 we introduce the symmetric random jump rate model (SRJM) and derive the response and Green's functions. In Section 3 we study the VACF and the current noise and in Section 4 the noise in the D fluctuations, and find that Eq. (1.2) is satisfied. The equivalence between Burnett functions and D fluctuations is shown in Section 5. In Section 6 the response function is calculated for weak and strong disorder, and applied in Section 7 to calculate spectral densities for strong disorder, showing behavior close to $1/f$ noise. In Section 8 we draw some conclusions.

2. RANDOM JUMP RATE MODEL

The model of interest here is a random walk or a hopping model on a random lattice with site disorder. The lattice is a d dimensional hypercubic lattice with lattice distance l and with M sites. On the lattice there are N independent hopping particles, each carrying a charge e. Only hops between nearest-neighbor sites are allowed. Let $w_{\mathbf{n}} = 1/\tau_{\mathbf{n}}$ be the jump rate from a given site $\mathbf{n}$ to the nearest-neighbor site $\mathbf{n}+\mathbf{b}$, and $\tau_{\mathbf{n}}/C$ the waiting time at site $\mathbf{n}$, where $C=2d$ is the coordination number of the lattice. Here $w_{\mathbf{n}}$ and $\tau_{\mathbf{n}}$ are random variables with a site-independent probability distribution, $\bar{\rho}(w)\, dw = \rho(\tau)\, d\tau$. If a uniform electric field E is applied along the positive x axis, a particle gains an amount of energy $2\varepsilon = eEl/kT$ (measured in thermal units), when hopping in the direction of the field, and its jump rate from site $\mathbf{n}$ to its nearest-neighbor site $\mathbf{n}+\mathbf{b}$ is given by $w_{\mathbf{n}} \exp(\varepsilon b_x)$.

Let $P_{\mathbf{n}}(t)$ be the probability that the hopping particle is at position $\mathbf{r}=\mathbf{n}l$ at time t. It satisfies the following master equation:

$$\dot{P}_{\mathbf{n}}=\sum_{\mathbf{b}} [w_{\mathbf{n}+\mathbf{b}} e^{-\varepsilon b_x} P_{\mathbf{n}+\mathbf{b}} - w_{\mathbf{n}} e^{\varepsilon b_x} P_{\mathbf{n}}] \equiv -(\Phi W P)_{\mathbf{n}} \tag{2.1}$$

where $\mathbf{b}$ runs over all nearest-neighbor sites. Here the operator Φ is defined as

$$\Phi=\sum_{\mathbf{b}} e^{\varepsilon b_x}[1-\mathfrak{E}_{\mathbf{b}}]=\sum_{\mathbf{b}} (e^{\varepsilon b_x}-\bar{e}^{\varepsilon b_x}\mathfrak{E}_{\mathbf{b}}) \tag{2.2}$$

where the shift operator $\mathfrak{E}_{\mathbf{b}}$ acts as $\mathfrak{E}_{\mathbf{b}} f(\mathbf{n})=f(\mathbf{n}+\mathbf{b})$.

The jump rate matrix is diagonal in coordinate representation with elements $W_{\mathbf{nm}}=w_{\mathbf{n}}\delta_{\mathbf{nm}}$, and the master equation has the stationary solution $\sim 1/w_{\mathbf{n}} \sim \tau_{\mathbf{n}}$, normalized as

$$P_{\mathbf{n}}(\infty)=\tau_{\mathbf{n}} \Big/ \sum_{\mathbf{m}} \tau_{\mathbf{m}} = v\tau_{\mathbf{n}}/M \tag{2.3}$$

Because the number of lattice points M is sufficiently large, we are allowed to identify the site average with the average $\langle \cdots \rangle$ over the distribution $\rho(\tau)$, so that

$$1/v \equiv \langle \tau \rangle = (1/M) \sum_{\mathbf{m}} \tau_{\mathbf{m}} \tag{2.4}$$

From here on the brackets $\langle \cdots \rangle$ denote an average over the random variables $\{\tau_{\mathbf{n}}\}$ or $\{w_{\mathbf{n}}\}$.

In the present model Eq. (2.3) is taken as the *initial distribution*. Hence the one-time distribution $P_{\mathbf{n}}(t)$ is always the stationary distribution (2.3). This steady state sustains a *drift velocity*

$$\begin{aligned} \langle v_x \rangle &\equiv l \sum_{\mathbf{n}} n_x \langle \dot{P}_{\mathbf{n}} \rangle \\ &= l \sum_{\mathbf{n}} \Big(\sum_{\mathbf{b}} b_x e^{\varepsilon b_x} \Big) w_{\mathbf{n}} P_{\mathbf{n}}(t) = 2vl \sinh \varepsilon \end{aligned} \tag{2.5}$$

as follows from (2.1) and (2.3), and corresponds to a steady current $I = eN\langle v_x \rangle / L$, where L is the length of the sample in the direction of the field. For small fields one recovers Ohm's law $I = V/R$ with a *resistance* R and *mobility* μ, defined as

$$\begin{aligned} R &= L^2/(e^2 \mu N) = kTL^2/(e^2 v l^2 N) \\ \mu &\equiv \langle v_x \rangle/(eE) = vl^2/kT \end{aligned} \tag{2.6}$$

For studying the fluctuations in the current we follow the method in Ref. 21 and introduce the response function

$$\mathfrak{F}(\mathbf{q}, z; \varepsilon) = \int_0^\infty dt\, e^{-zt} \sum_{\mathbf{n},\mathbf{m}} e^{-i\mathbf{q}(\mathbf{n}-\mathbf{m})l} \langle P(\mathbf{n}, t; \mathbf{m}, 0)\rangle \tag{2.7}$$

It is the Laplace transform of the generating function for walks where the particle has moved a distance $(\mathbf{n}-\mathbf{m})$ from its starting point $\mathbf{m}$. Since the master equation (2.1) describes a Markov process, the two-point distribution $P(\mathbf{n}t; \mathbf{m}0)$ can be expressed in the conditional probability $P(\mathbf{n}t \mid \mathbf{m}0)$, i.e.,

$$P(\mathbf{n}t; \mathbf{m}0) = P(\mathbf{n}t \mid \mathbf{m}0)\, P_{\mathbf{m}}(0) = P(\mathbf{n}t \mid \mathbf{m}0)\, P_{\mathbf{m}}(\infty) \tag{2.8}$$

Thus, we deduce from (2.1) and (2.7)

$$\mathfrak{F}(\mathbf{q}, z; \varepsilon) = v\langle (z + \Phi W)^{-1}\, T\rangle_{\mathbf{qq}} \tag{2.9}$$

where the matrix of waiting times $T \equiv W^{-1}$ is diagonal in coordinate space with elements $T_{\mathbf{nn}} = \tau_{\mathbf{n}} \equiv 1/w_{\mathbf{n}}$. Denoting the matrix $(z + \Phi W)^{-1}\, T$ by A we observe that the average matrix $\langle A_{nm}\rangle$ is translation invariant, because the distribution of the randomness itself is translation invariant. Thus A is diagonal in Fourier space, i.e., $\langle A\rangle_{\mathbf{qq}'} = \delta_{\mathbf{qq}'}\langle A\rangle_{\mathbf{qq}}$ with

$$\langle A\rangle_{\mathbf{qq}} = N^{-1} \sum_{\mathbf{nm}} e^{-i\mathbf{q}\cdot(\mathbf{n}-\mathbf{m})l} \langle A_{\mathbf{nm}}\rangle \tag{2.10}$$

Subscripts $\mathbf{q}$ always refer to the Fourier representation, and subscripts $\mathbf{n}$, $\mathbf{m}$ to the coordinate representation.

Using the relation

$$(z + \Phi W)^{-1} = Tz^{-1} - \Phi z^{-2} + \Phi(zT + \Phi)^{-1}\, \Phi z^{-2}$$

the response function (2.9) may be rearranged as

$$\mathfrak{F}(\mathbf{q}, z; \varepsilon) = z^{-1} - z^{-2} v\Phi(q) + z^{-2} v\Phi^2(q)\, G(\mathbf{q}, z; \varepsilon) \tag{2.11}$$

where the detailed dependence of $\mathfrak{F}$ on the randomness is contained in the Green's function:

$$G(\mathbf{q}, z; \varepsilon) = \langle (zT + \Phi)^{-1}\rangle_{\mathbf{qq}} \tag{2.12}$$

This quantity will be calculated explicitly in Sections 6 and 7. The matrix Φ, defined in (2.2), is diagonal in Fourier space and has components

$$\Phi(\mathbf{q}) = 2\cosh\varepsilon - 2\cosh(\varepsilon - iq_x l) + 2\sum_{\alpha=y}^{d}(1-\cos q_\alpha l)$$
$$\simeq 2iq_x l\sinh\varepsilon + (q_x l)^2(\cosh\varepsilon - 1) + q^2 l^2 + \cdots \tag{2.13}$$

The last equality applies to small q values.

3. VACF AND CURRENT NOISE

The response function $\mathfrak{F}(\mathbf{q}, z; \varepsilon)$ contains all macroscopic quantities of interest, e.g., its $\mathbf{q}$ expansion generates the (Laplace-transformed) moments of displacement. These moments, in turn, can be expressed in terms of the Green's function $G(\mathbf{q}, z; \varepsilon)$ with the help of (2.11).

We start with the moments of displacement

$$\left(i\frac{\partial}{\partial q_\alpha}\right)^m \mathfrak{F}(\mathbf{q}, z; \varepsilon)\Big|_{\mathbf{q}=\mathbf{0}} = \mathfrak{L}_z\langle[\Delta r_\alpha(t)]^m\rangle \tag{3.1}$$

where $\mathfrak{L}_z$ denotes the Laplace transform, and

$$\Delta r_\alpha(t) = l[n_\alpha(t) - n_\alpha(0)] \tag{3.2}$$

For instance, if $\varepsilon = 0$ one finds immediately from (3.1) and (2.11) that $\langle[\Delta r_\alpha(t)]^2\rangle = 2Dt$ with $D = \nu l^2$.

Lattice equivalents of velocity correlation functions can be introduced as derivatives of these moments, writing formally $\Delta r_\alpha(t) = \int_0^t v_\alpha(\tau)\, d\tau$:

$$\begin{aligned}\langle v_\alpha\rangle &= (d/dt)\langle\Delta r_\alpha(t)\rangle \\ \langle v_\alpha(t)\, v_\alpha(0)\rangle &= \tfrac{1}{2}(d/dt)^2\langle[\Delta r_\alpha(t)]^2\rangle\end{aligned} \tag{3.3}$$

A convenient way to express these correlation functions (1.6) in $\mathfrak{F}(q, z; \varepsilon)$ and $G(q, z; \varepsilon)$ is for the drift velocity,

$$\langle v_\alpha\rangle = -i\frac{\partial}{\partial q_\alpha}[\mathfrak{F}(\mathbf{q}, z; \varepsilon)]^{-1}\Big|_{\mathbf{q}=\mathbf{0}} = 2\nu l(\sinh\varepsilon)\,\delta_{\alpha x} \tag{3.4}$$

in agreement with (2.5), and for the VACF's,

$$\hat{\varphi}_{\|}(z; \varepsilon) = \frac{1}{2}\left(\frac{\partial}{\partial q_x}\right)^2[\mathfrak{F}(\mathbf{q}, z; \varepsilon)]^{-1}\Big|_{\mathbf{q}=\mathbf{0}}$$
$$= \nu l^2\cosh\varepsilon + (\langle v_x\rangle^2/\nu)[G(\mathbf{0}, z; \varepsilon) - \nu/z] \tag{3.5}$$

and

$$\hat{\varphi}_\perp(z;\varepsilon) = \nu l^2 \equiv D \tag{3.6}$$

In absence of a bias field $\langle v_\alpha \rangle = 0$, and $\hat{\varphi}_\perp(z, 0) \equiv \hat{\varphi}_0(z) = D$ is independent of z, where the VACF behaves as

$$\varphi_0(t) = \langle v_\alpha(t)\, v_\alpha(0) \rangle = D\delta(t) \tag{3.7}$$

for $\alpha = x, y,..., d$, as first shown by Haus *et al.*[22] We also note that the first term in (3.5) contributes a term $(D \cosh \varepsilon)\, \delta(t)$ to $\varphi_{||}(t;\varepsilon)$, so that in the limit $t \to 0^+$ one has the relation

$$\varphi_{||}(0^+;\varepsilon) = \langle v_x \rangle^2 (\nu_\infty - \nu)/\nu \tag{3.8}$$

In deriving this result we used (2.12) and (2.10) to write

$$\lim_{z \to \infty} zG(\mathbf{0}, z; \varepsilon) = \langle T^{-1} \rangle_{\mathbf{qq}} |_{\mathbf{q}=\mathbf{0}} = \langle \tau^{-1} \rangle \equiv \nu_\infty \tag{3.9}$$

where $\nu_\infty = \langle 1/\tau \rangle$ is the bare, short time or high-frequency average jump rate, which approaches a plateau value in this model. It is proportional to an Enskog-type (i.e., short time) diffusion coefficient $D_\infty = \nu_\infty l^2$.[8,20,23] The actual diffusion coefficient $D = \nu l^2$ with $\nu = 1/\langle \tau \rangle$ is comparable to the mode-coupling diffusion coefficient, which is renormalized by the fluctuations.[23]

Next we determine the spectral density of current fluctuations, which is related to the VACF in (3.5) on account of (1.5), and we find

$$S_I(\omega) = (4kT/R) \cosh \varepsilon + (4I^2/N\nu)\, \Re e G(\mathbf{0}, i\omega; \varepsilon) \tag{3.10}$$

Here $I = (eN/L)\langle v_x \rangle$ is the steady nonvanishing current, which reduces for small fields ($\varepsilon = elE/2kT$), where $\langle v_x \rangle \simeq 2\nu l\varepsilon$, to Ohm's law with a resistance $R = kTL^2/(e^2DN)$. The current noise (3.10) has a white background, which reduces for small fields to the standard Johnson noise $4kT/R$. The second term is the excess current noise $S_I^{\text{exc}}(\omega)$ in the presence of a field, and is proportional to I^2.

The transverse current noise,

$$S_\perp(\omega) = (4e^2N/L_y^2)\, \Re e \hat{\varphi}_\perp(i\omega;\varepsilon) = 4kT/R_y \tag{3.11}$$

contains only Johnson noise, where $R_y = kTL_y^2/(e^2DN)$ is the resistance of the sample for a current flowing in the y direction, where the sample length is L_y.

Another quantity of interest is the total excess current noise, which follows from (3.8) as

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_I^{\text{exc}}(\omega) = (e^2N/L^2)\, \varphi_{\|}(0^+; \varepsilon) = (I^2/N)(v_\infty - v)/v \qquad (3.12)$$

This quantity, and therefore the excess noise itself, vanishes on a uniform lattice without disorder, as it should.

4. MOBILITY AND RESISTANCE FLUCTUATIONS

In general, fluctuations in the resistance $R = L^2/(e^2\mu N)$ in (2.6) result from fluctuations δN in the number of charge carriers and from fluctuations $\delta\mu = \delta D/kT$ in the mobility or diffusion coefficient. In the present model $N \sim \sum_{\mathbf{n}} P_{\mathbf{n}}$ is a constant and does not fluctuate. In this section we therefore consider D or μ fluctuations, and define a fluctuating diffusion coefficient (in the absence of an electric field) as

$$D(t, \tau) = \tfrac{1}{2}\partial_t (x(t) - x(\tau))^2 = \tfrac{1}{2} l^2 \partial_t (n_x(t) - n_x(\tau))^2 \qquad (4.1)$$

with the corresponding averages:

$$\langle D(t, \tau)\rangle = \tfrac{1}{2} l^2 \left\langle \sum_{\mathbf{n},\mathbf{m}} (n_x - m_x)^2\, \partial_t P(\mathbf{n}t; \mathbf{m}\tau) \right\rangle \qquad (4.2)$$

$$\langle D(t, \tau)\, D(t', \tau')\rangle = \tfrac{1}{4} l^4 \left\langle \sum_{\substack{\mathbf{n},\mathbf{m} \\ \mathbf{n}',\mathbf{m}'}} (n_x - m_x)^2 (n'_x - m'_x)^2 \times \partial_t \partial_{t'} P(\mathbf{n}t; \mathbf{m}\tau; \mathbf{n}'t'; \mathbf{m}'\tau') \right\rangle \qquad (4.3)$$

We could set $\tau' = 0$ because of stationarity. The D fluctuation in (4.3) is a fourth-rank tensor with cubic symmetry, and it is straightforward to extend the present calculation to the $(xxyy)$ element. Since the master equation (2.1) for a fixed set of jump rates $\{w_{\mathbf{n}}\}$ describes a Markov process, two-, three-, and four-point distributions can be expressed as product of conditional probabilities, as in (2.8), e.g., $P(\mathbf{n}t; \mathbf{n}'t'; \mathbf{m}0) = P(\mathbf{n}t \mid \mathbf{n}'t')\, P(\mathbf{n}'t' \mid \mathbf{m}0)\, P_{\mathbf{m}}(\infty)$ when $t > t' > 0$.

The conditional probability obeys the initial condition $P(\mathbf{n}, 0 \mid \mathbf{m}, 0) = \delta_{\mathbf{nm}}$ and satisfies the forward and backward master equation:

$$\begin{aligned} \partial_t P(\mathbf{n}t \mid \mathbf{n}'t') &= -\partial_{t'} P(\mathbf{n}t \mid \mathbf{n}'t') \\ &= -\Phi(\mathbf{n})\, w_{\mathbf{n}} P(\mathbf{n}t \mid \mathbf{n}'t') = -w_{\mathbf{n}'} \Phi(\mathbf{n}')\, P(\mathbf{n}t \mid \mathbf{n}'t') \end{aligned} \qquad (4.4)$$

In the field free case $-\Phi(\mathbf{n}) = \sum_\alpha (\mathfrak{E}_\alpha + \mathfrak{E}_\alpha^{-1} - 2)$ with $\alpha = x, y,..., d$ is the discrete Laplacian with $\mathfrak{E}_\alpha f(n_\beta) = f(n_\beta + \delta_{\alpha\beta})$. Using the forward master equation in (4.3), performing a partial summation and employing the relation $\Phi(\mathbf{n})(n_x - m_x)^2 = -2$, it follows immediately that

$$\begin{aligned} \langle D(t, \tau) \rangle &= l^2 \left\langle \sum_{\mathbf{n},\mathbf{m}} w_\mathbf{n} P(\mathbf{n}t; \mathbf{m}\tau) \right\rangle \\ &= l^2 \left\langle \sum_\mathbf{n} w_\mathbf{n} P_\mathbf{n}(t) \right\rangle = D \end{aligned} \tag{4.5}$$

In the last equality we used the relation $P_\mathbf{n}(t) = P_\mathbf{n}(\infty)$ as given in (2.3) with $D = vl^2$. In general $\langle D(t, \tau) \rangle$ may be expressed as an integral over the VACF on account of (4.1) and (3.3), i.e.,

$$\langle D(t, \tau) \rangle = \int_\tau^t ds \langle v_x(t)\, v_x(s) \rangle = \int_0^{t-\tau} ds \langle v_x(s)\, v_x(0) \rangle \tag{4.6}$$

It depends only on $(t - \tau)$, since the average is over a stationary initial distribution. It is a special feature of the isotropy of the jump rates in our model that $\langle D(t, \tau) \rangle$ is actually time independent as follows also from (4.6) and (3.7).

Next, consider the four-point correlation in (4.3), for which we find by the same manipulation (for $t > \tau > t' > \tau'$)

$$\langle D(t, \tau)\, D(t', \tau') \rangle = \tfrac{1}{2} l^4 \left\langle \sum_{\substack{\mathbf{nm} \\ \mathbf{n'm'}}} w_\mathbf{n} (n'_x - m'_x)^2\, \partial_{t'} P(\mathbf{n}t; \mathbf{m}\tau; \mathbf{n't'}; \mathbf{m'}\tau') \right\rangle \tag{4.7a}$$

By carrying out the $\mathbf{m}$ summation in Eq. (4.7a) we see that the left-hand side of (4.7) is independent of τ for all $0 < \tau < t$. Next,

$$\begin{aligned} &\langle D(t, \tau)\, D(t', \tau') \rangle \\ &= \tfrac{1}{2} l^4 \left\langle \sum_{\mathbf{n},\mathbf{n}',\mathbf{m}} w_\mathbf{n} (n'_x - m_x)^2\, \partial_{t'} P(\mathbf{n}t \mid \mathbf{n}'t')\, P(\mathbf{n}'t' \mid \mathbf{m}\tau')\, P_\mathbf{m}(\infty) \right\rangle \\ &= \tfrac{1}{2} l^4 \left\langle \sum_{\mathbf{n},\mathbf{n}',\mathbf{m}} w_\mathbf{n} P(\mathbf{n}t \mid \mathbf{n}'t') [\Phi(\mathbf{n}'), (n'_x - m_x)^2]\, w_{\mathbf{n}'} P(\mathbf{n}'t' \mid \mathbf{m}\tau')\, P_\mathbf{m}(\infty) \right\rangle \end{aligned} \tag{4.7b}$$

In the next step we want to show that the left-hand side of (4.7b) is independent of τ'. To that purpose we evaluate the commutator on the third line of (4.7b) to find

$$[\Phi(\mathbf{n}), (n_x - m_x)^2] = (\mathfrak{E}_x + \mathfrak{E}_x^{-1}) - 2(\mathfrak{E}_x - \mathfrak{E}_x^{-1})(n_x - m_x) \tag{4.8}$$

and use the following identities:

$$\begin{gathered} \sum_{\mathbf{m}} (n_x - m_x)\, P(\mathbf{n}t \mid \mathbf{m}\tau')\, P_{\mathbf{m}}(\infty) = 0 \\ \sum_{\mathbf{m}} w_{\mathbf{n}} P(\mathbf{n}t \mid \mathbf{m}\tau')\, P_{\mathbf{m}}(\infty) = w_{\mathbf{n}} P_{\mathbf{n}}(\infty) = v/M \end{gathered} \tag{4.9}$$

The first equality holds at $t = \tau'$ and its derivative vanishes at all times; the second line of (4.9) follows from the stationarity of $P_{\mathbf{m}}(\infty)$.

After inserting (4.8) into (4.7b) and summing over $\mathbf{m}$, one can show with the help of the first identity in (4.9) that the second term in (4.8) gives a vanishing contribution. The second identity in (4.9) shows that $(\mathfrak{E}_x + \mathfrak{E}_x^{-1})$ in (4.8) can effectively be replaced by 2. Performing the $\mathbf{m}$ summation one finally obtains

$$\begin{aligned} \langle D(t, \tau)\, D(t', \tau') \rangle &= l^4 \left\langle \sum_{\mathbf{n}\mathbf{n}'} w_{\mathbf{n}} w_{\mathbf{n}'} P(\mathbf{n}t; \mathbf{n}'t') \right\rangle \\ &= (v l^4/M) \left\langle \sum_{\mathbf{n}\mathbf{n}'} w_{\mathbf{n}} P(\mathbf{n}t \mid \mathbf{n}'t') \right\rangle \equiv C_D(t - t') + D^2 \end{aligned} \tag{4.10}$$

In general the four-point correlation function involved in the D fluctuations, will depend on three time differences. Owing to the isotropy of the present model, however, it is independent of τ and τ', and depends only on the time difference $t - t'$.

The Laplace transform of $C_D(t)$ can be obtained from (4.10) and (4.4) and yields

$$\begin{aligned} \hat{C}_D(z) &= (v l^4/M) \left\langle \sum_{\mathbf{n}\mathbf{n}'} [W(z + \Phi W)^{-1}]_{\mathbf{n}\mathbf{n}'} \right\rangle - v l^4/z \\ &= v l^4 [G(\mathbf{0}; z; \varepsilon = 0) - v/z] \end{aligned} \tag{4.11}$$

where we have used (2.12) and the matrix relation $W(z + \Phi W)^{-1} = (zT + \Phi)^{-1}$ with $T = W^{-1}$. On account of (2.10) the double sum in (4.11) represents the matrix element $(\mathbf{q}, \mathbf{q}') = (\mathbf{0}, \mathbf{0})$ of this matrix in Fourier space. Thus we have expressed the fluctuations in the diffusion coefficient of one given particle in terms of the response function $G(\mathbf{q}, z; \varepsilon)$ for the SRJM, and the spectral density of D fluctuation is given by (1.11).

It is also instructive to consider the mobility fluctuations explicitly, and we define a fluctuating mobility as

$$\mu(t)=\frac{1}{N}\sum_{\mathbf{n}} N_{\mathbf{n}}(t)\,\mu_{\mathbf{n}} \tag{4.12}$$

where $N_{\mathbf{n}}(t)$ is the number of charge carriers at site $\mathbf{n}$ at time t, N their total number, and $\mu_{\mathbf{n}}=l^2 w_{\mathbf{n}}/kT$ the local (frozen-in) mobility. The mobility, averaged over random walks but not over the random variables $w_{\mathbf{n}}$ (indicated by $\langle\cdots\rangle_{\mathrm{RW}}$) is by definition related to the one-time probability $P_{\mathbf{n}}(t)$:

$$\langle\mu(t)\rangle_{\mathrm{RW}}=\frac{1}{N}\sum_{\mathbf{n}} P_{\mathbf{n}}(t)\,\mu_{\mathbf{n}}=l^2 v/kT \tag{4.13}$$

where we used the stationarity condition $P_{\mathbf{n}}(t)=P_{\mathbf{n}}(\infty)$ and (2.4). In the present model (4.13) is independent of $\{w_{\mathbf{n}}\}$.

For calculating the two-time mobility correlation we have to keep track on the identity of the particles. Let $N_{\mathbf{n}}^i(t)$ be equal to unity if the ith particle is at site $\mathbf{n}$ at time t, and zero elsewhere. Then it follows that $N_{\mathbf{n}}=\sum_i N_{\mathbf{n}}^i$, and owing to the independence of the particles one has

$$\langle N_{\mathbf{n}}(t)\,N_{\mathbf{n}'}(t')\rangle_{\mathrm{RW}}=N(N-1)\,P_{\mathbf{n}}(t)\,P_{\mathbf{n}'}(t')+NP(\mathbf{n},t;\mathbf{n}',t') \tag{4.14}$$

Using (4.14) we obtain for the two-point mobility correlation function, averaged over the static disorder,

$$\langle\langle\mu(t)\,\mu(t')\rangle_{\mathrm{RW}}-\langle\mu\rangle_{\mathrm{RW}}^2\rangle=\frac{1}{N}C_\mu(t-t') \tag{4.15}$$

where

$$C_\mu(t-t')=\left\langle\sum_{\mathbf{n},\mathbf{n}'}\mu_{\mathbf{n}}\mu_{\mathbf{n}'}[P(\mathbf{n}t;\mathbf{n}'t')-P(\mathbf{n}t)\,P(\mathbf{n}'t')]\right\rangle \tag{4.16}$$

The factor $(1/N)$ in (4.15) arises from the independence of the hopping particles. By comparison with (4.10a) and (4.14) we can identify

$$C_\mu(t)=(kT)^{-2}\,C_D(t) \tag{4.17}$$

The main purpose of this section was to find the connection between the excess current noise $S_I^{\mathrm{exc}}(\omega)$ and the resistance or mobility fluctuations. Since fluctuations in resistance R, mobility μ, and coefficient D are related as $\delta R = -(R/\mu)\,\delta\mu = -(R/D)\,\delta D$, the spectral density of resistance fluctuations is

$$S_R(\omega)/R^2 = S_\mu(\omega)/\mu^2 = (4/N\nu)\,\Re e G(\mathbf{0}, i\omega; \varepsilon = 0) \tag{4.18}$$

Also the excess current noise $S_I^{\text{exc}}(\omega)$, given by the second term in (3.10) contains $G(\mathbf{0}; i\omega; \varepsilon \neq 0)$. We will show in the next section that the field dependence of $G(\mathbf{0}, z; \varepsilon)$ is negligible for small ε for dimensions $d > 2$, and of minor importance for $d = 1$ and $d = 2$. For the SRJM in $d \geqslant 3$ we have, therefore, shown that for small I

$$S_I^{\text{exc}}(\omega)/I^2 = S_R(\omega)/R^2 = S_\mu(\omega)/\mu^2 \tag{4.19}$$

This equation is valid for *all frequencies* and *arbitrary disorder*. The integrated spectral density of resistance fluctuations can be deduced from (4.11) and (3.9) as

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_R(\omega) = \frac{R^2}{N} \frac{\nu_\infty - \nu}{\nu} \tag{4.20}$$

This is in complete agreement with the integrated excess current noise in (3.12) in the presence of a field.

5. BURNETT CORRELATION FUNCTION AND RESISTANCE FLUCTUATIONS

In the present section the electric field is again set equal to zero. For the present model there exists in this case a relationship between the correlation function of resistance fluctuations and the Burnett correlation function $\beta(t)$ appears in generalized hydrodynamics and is defined as the second derivative of the fourth cumulant:[(24)]

$$\begin{aligned}\beta(t) &= \frac{1}{24}\left(\frac{\partial}{\partial t}\right)^2 \{\langle [\Delta x(t)]^4\rangle - 3\langle [\Delta x(t)]^2\rangle^2\} \\ &= \int_0^t d\tau \int_0^\tau d\tau' \langle\langle v_x(t)\, v_x(\tau)\, v_x(\tau')\, v_x(0)\rangle\rangle \end{aligned} \tag{5.1}$$

where $\langle\langle\cdots\rangle\rangle$ is defined in (1.14). The Burnett correlation function is actually a completely symmetric fourth-rank tensor with cubic symmetry, which has two independent elements $(xxxx)$ and $(xxyy)$. It is straightforward but rather lengthy to extend the present calculation to the component $(xxyy)$. The result will not be given here.

The Laplace transform $\hat{\beta}(z)$ can be expressed in the response function, as shown in Appendix A:

$$\hat{\beta}(z) = \frac{-1}{24}\left(\frac{\partial}{\partial q_x}\right)^4 [\mathfrak{F}(\mathbf{q}, z; 0)]^{-1}\Big|_{\mathbf{q}=\mathbf{0}} \tag{5.2}$$

or with the help of (2.11) in the Green's function:

$$\hat{\beta}(z) = \frac{1}{12} v l^4 + v l^4 \left[G(\mathbf{0}, z; 0) - \frac{v}{z} \right] \tag{5.3}$$

The first term in (5.3) contributes a term $(vl^4/12)\,\delta(t)$ to $\beta(t)$ and by the same arguments as given in (3.8) and (3.9), one finds that $\beta(0^+) = v(v_\infty - v)\,l^4$.

Comparison of (4.11) and (5.3) shows that the Burnett correlation function $\beta(t)$ for $t > 0$ equals the correlation function $C_D(t)$ of D fluctuations or resistance fluctuations.

In order to test for the present model the assumptions made by Stanton and Nelkin regarding the behavior of the four-point current correlation function, we consider the four-point velocity correlation function in more detail. To do so we write the D fluctuation in (4.3) with the help of (4.1) as

$$\langle D(t, \tau)\, D(t', \tau') \rangle = \int_\tau^t ds \int_{\tau'}^{t'} ds' \langle v_x(t)\, v_x(s)\, v_x(t')\, v_x(s') \rangle \tag{5.4}$$

Because the left-hand side of (5.4) is independent of τ and τ' in the interval $t > \tau > t' > \tau'$ on account of (4.10), the integrand of (5.4) in the corresponding interval is $\delta(t-s)\,\delta(t'-s')[C_D(t-t') + D^2]$. For different time orderings one has a similar expression proportional to $\delta(t-s')\,\delta(t'-s)$ and to $\delta(t-t')\,\delta(s-s')$.

The result of (5.3) that $\beta(t) \simeq (vl^4/12)\,\delta(t) + \beta(0^+)$ for $t \downarrow 0$, shows that the four-point correlation function in (5.1) has an additional singular short-time contribution when all four time arguments are close together, so that finally the four-point velocity cumulant becomes

$$\begin{aligned} &\langle\langle v_x(t)\, v_x(\tau)\, v_x(t')\, v_x(\tau') \rangle\rangle \\ &\quad = \frac{1}{12} v l^4 \delta(t-t')\,\delta(t-\tau)\,\delta(t'-\tau') \\ &\qquad + [\delta(t-\tau)\,\delta(t'-\tau') + \delta(t-\tau')\,\delta(t'-\tau)]\, C_D(t-t') \\ &\qquad + \delta(t-t')\,\delta(\tau-\tau')\, C_D(t-\tau) \end{aligned} \tag{5.5}$$

This shows that two pairs of velocities are only *delta correlated* in the SRJM, and verifies one of the assumptions of Stanton and Nelkin[9] that the four-point velocity correlation function is only nonvanishing if the time differences $t-\tau$ and $t'-\tau'$ (or permutations thereof) do not exceed some *microscopic correlation time.*

6. FLUCTUATION AND t-MATRIX EXPANSION

6.1. Weak Disorder

In the previous part of the paper we have shown that the excess current and resistance noise are equal for our model, and also that the Burnett correlation function equals the correlation function of the fluctuations in the diffusion coefficient. This was done without any specification of the randomness in the system, i.e., the distribution $\rho(\tau)$. In order to make explicit calculations of the low-frequency behavior of the spectral densities one has to specify the disorder in the system, i.e., the distribution of waiting times $\rho(\tau)$. We want to distinguish weak and strong disorder, and start with the case of *weak disorder*, where a sufficiently (to be specified later) large number of moments $\langle \tau^n \rangle$ exists.

In order to study the low-frequency behavior of $S_I(\omega)$ and $S_R(\omega)$ in (3.10) and (4.12) one needs the small-z behavior of the Green's function $G(\mathbf{q}, z; \varepsilon) = \langle [zT + \Phi]^{-1} \rangle_{\mathbf{qq}}$ at $\mathbf{q} = \mathbf{0}$. Following the method introduced in Ref. 21 for the d-dimensional SRJM in the field free case this can be calculated by making an expansion in powers of the fluctuation matrix Δ_0, defined as

$$T = \langle \tau \rangle \mathbb{1} + \Delta_0 \tag{6.1a}$$

or

$$(\Delta_0)_{\mathbf{nm}} = \delta_{\mathbf{nm}}(\tau_{\mathbf{n}} - \langle \tau \rangle) \tag{6.1b}$$

where Δ_0 is diagonal in coordinate space. Next we introduce the Green's function for the uniform average lattice

$$g_0(\mathbf{q}, z; \varepsilon) = [z\langle \tau \rangle + \Phi(\mathbf{q})]^{-1} \tag{6.2}$$

and the corresponding single-site Green's function

$$\Psi_0(z; \varepsilon) = (2\pi)^{-d} \int_{1\mathrm{BZ}} d\mathbf{q} [z\langle \tau \rangle + \Phi(\mathbf{q})]^{-1} \tag{6.3}$$

where the integration extends over the first Brillouin zone. Next, we expand $G(\mathbf{q}, z; \varepsilon)$ in powers of Δ_0 as

$$G(\mathbf{q}, z; \varepsilon) = \langle g_0 + z^2 g_0 \Delta_0 g_0 \Delta_0 g_0 - z^3 g_0 \Delta_0 g_0 \Delta_0 g_0 \Delta_0 g_0 + \cdots \rangle_{\mathbf{qq}} \quad (6.4a)$$

$$= g_0 + \kappa_2 (z g_0)^2 \Psi_0(z; \varepsilon) + \cdots \quad (6.4b)$$

Here we have used the relations $\langle \Delta_0 \rangle = 0$ and $\langle \Delta_0 g_0 \Delta_0 \rangle_{\mathbf{qq}} = \kappa_2 \Psi_0(z; \varepsilon)$ as follow form (2.10) and (6.1), and introduced the variance $\langle \tau^2 \rangle - \langle \tau \rangle^2 \equiv \kappa_2$. The terms in (6.4b) give the dominant small-z behavior as shown in Ref. 21.

As we are mainly interested in *small fields*, we evaluate $\Psi_0(z; \varepsilon)$ only for small ε and small z:

$$\Psi_0(z; \varepsilon) = \begin{cases} \frac{1}{2}(z\langle \tau \rangle + \varepsilon^2)^{-1/2} & (d=1) \\ -(4\pi)^{-1} \log[z\langle \tau \rangle + \varepsilon^2] & (d=2) \\ \Psi_0(0; 0) - (4\pi)^{-1}(z\langle \tau \rangle + \varepsilon^2)^{1/2} & (d=3) \end{cases} \quad (6.5)$$

On account of (3.10), (4.12), and (6.4) the dominant low-frequency behavior of $S_I(\omega)$ and $S_R(\omega)$ is proportional to $\Re e G(\mathbf{0}, i\omega; \varepsilon) \sim \kappa_2 \Re e \Psi_0(i\omega; \varepsilon)$. For a full discussion it is important to distinguish an intermediate-frequency regime dominated by diffusion, where $\varepsilon^2 \ll z\langle \tau \rangle \ll 1$ and a small-z regime, where $z\langle \tau \rangle \ll \varepsilon^2$, dominated by the linear drift.

In the *diffusion-dominated regime* the current noise has a dominant small ω singularity

$$S_I(\omega) \sim (I^2/N)(\langle \tau^2 \rangle - \langle \tau \rangle^2)\, \omega^{(1/2)d-1} \quad (6.6)$$

for d even, and multiplied with $\log(1/\omega)$ for d odd, with a coefficient that can be easily calculated. This result represents the well-known *long time tail* of the Burnett function in the absence of a field, as derived in Refs. 13 and 21. In the drift-dominated regime the excess current noise, $S_I^{\text{exc}}(\omega) \sim |\varepsilon|^{d-2}$ for d odd and $S_I^{\text{exc}}(\omega) \sim |\varepsilon|^{d-2} \log(1/\varepsilon)$ for d even, is white and gives only a small contribution to the Johnson noise. The one-dimensional results in the presence of a field were obtained before by Lehr *et al.*[14]

The above results, referring to the case of weak disorder, require the existence of, at least, $\kappa_2 \sim \langle \tau^2 \rangle - \langle \tau \rangle^2$. This condition is by no means sufficient as will be shown in Section 7.

In concluding this section we observe that the noise (6.6), $S(\omega) \sim \sqrt{\omega}$, found in the weakly disordered three-dimensional SRJM, is far from the experimentally observed noise $S(\omega) \sim 1/\omega$. We therefore investigate next strongly disordered systems.

6.2. Strong Disorder

In the case of *strong disorder* κ_2 is divergent, indicating that $G(0, i\omega; \varepsilon) = \kappa_2 \Psi_0(i\omega; \varepsilon)$ in (6.4b) becomes singular for all dimensions both in the diffusion- and in the drift-dominated regime.

It is interesting to know what type of waiting time distribution $\rho(\tau)$ gives rise to what type of singularity in $S_l(\omega)$ at low frequencies. The type of divergence in κ_2 depends on the large τ tail of $\rho(\tau)$, which may for example be represented as $\rho(\tau) \sim \tau^{-2-a}$. For $a \leqslant 1$ the variance diverges, corresponding to a case of *strong disorder*.

To perform the actual calculations the fluctuation expansion is useless, since each term $\langle (g_0 \Delta_0)^n \rangle_{\mathbf{qq}}$ $(l = 2, 3,...)$ in (6.4) diverges as $\langle \tau^n \rangle$. For a discussion of this divergence the coordinate representation, in which Δ_0 is a diagonal matrix, is more convenient that the Fourier representation. The dominant contributions come from those terms, where the labels $\mathbf{n}$ on all consecutive $\Delta_{\mathbf{nn}}$ are the same. Such terms can be resummed by a t-matrix resummation, which can be made self-consistent using the *effective medium approximation* (EMA) or hypernetted chain approximation. For the case of weak disorder this method correctly gives the dominant low-frequency behavior of this d-dimensional jump rate model, as shown in Refs. 21 and 23.

In the EMA one replaces the random matrix T by the sure matrix $\tau(z)\mathbb{1}$, where $\tau(z)$ is the ("frequency-dependent") waiting time of the effective medium. Then the EMA result for $G(\mathbf{q}, z; \varepsilon)$ is

$$g(\mathbf{q}, z; \varepsilon) = [z\tau(z) + \Phi(\mathbf{q})]^{-1} \tag{6.7}$$

The quantity $\tau(z)$ is determined from the EMA condition that the average t-matrix vanishes.[25] To work this out we introduce the new fluctuation matrix $\Delta = T - \tau(z)\mathbb{1}$, diagonal in coordinate space and expand $\tilde{G}_{\mathbf{kn}}(z) = (zT + \Phi)^{-1}_{\mathbf{kn}}$ in powers of Δ, where $\tilde{g}_{\mathbf{kn}} = [z\tau(z) + \Phi]^{-1}_{\mathbf{kn}}$ is the zeroth-order term. Next we perform a t-matrix resummation. The t matrix is diagonal an has elements

$$t_{\mathbf{nn}} = \Delta_{\mathbf{nn}}[1 + z\tilde{g}_{\mathbf{nn}}(z)\, \Delta_{\mathbf{nn}}]^{-1} \tag{6.8}$$

where $\tilde{g}_{\mathbf{kn}}(z)$ depends only on $|\mathbf{k} - \mathbf{n}|$, because of translational invariance of the effective medium. This yields the t-matrix expansion of the average Green's function

$$\begin{aligned} \langle \tilde{G}_{\mathbf{kn}} \rangle = {} & \tilde{g}_{\mathbf{kn}} - z \sum_{l} \tilde{g}_{\mathbf{k}l} \langle t_{ll} \rangle\, \tilde{g}_{l\mathbf{n}} + z^2 \sum_{l \neq \mathbf{m}}' \tilde{g}_{\mathbf{k}l} \langle t_{ll} \rangle\, \tilde{g}_{l\mathbf{m}} \langle t_{\mathbf{mm}} \rangle\, \tilde{g}_{\mathbf{mn}} \\ & - z^3 \sum_{l\mathbf{mm}'}' \langle \tilde{g}_{\mathbf{k}l} t_{ll} \tilde{g}_{l\mathbf{m}} t_{\mathbf{mm}} \tilde{g}_{\mathbf{mm}'} t_{\mathbf{m}'\mathbf{m}'} \tilde{g}_{\mathbf{m}'\mathbf{n}} \rangle \\ & + z^4 \sum_{ll'\mathbf{mm}'}' \langle \tilde{g}_{\mathbf{k}l} t_{ll} \tilde{g}_{ll'} t_{l'l'} \tilde{g}_{l'\mathbf{m}} t_{\mathbf{mm}} \tilde{g}_{\mathbf{mm}'} t_{\mathbf{m}'\mathbf{m}'} \tilde{g}_{\mathbf{m}'\mathbf{n}} \rangle + \cdots \end{aligned} \tag{6.9}$$

where the labels on two consecutive t matrices have to be different, as indicated by the prime. The EMA condition $\langle t_{\mathbf{nn}} \rangle = \langle t_{\mathbf{00}} \rangle = 0$ shows that the first nonvanishing t contribution to (6.9) has the structure $z^4 \tilde{g} \langle t_1 \tilde{g} t_2 \tilde{g} t_1 \tilde{g} t_2 \rangle \tilde{g}$. As long as this term is small compared to the EMA result, $\langle \tilde{G}_{\mathbf{kn}} \rangle = \tilde{g}_{\mathbf{kn}}$, in the relevant low-frequency regime, one can use the EMA to determine the dominant small-z behavior. For certain cases of strong disorder and dimensionality $d \leqslant 2$ the correction term can be of equal importance and then the EMA is only meaningfull in a small density expansion. This will be discussed later on.

The EMA condition $\langle t_{\mathbf{00}} \rangle = 0$ requires through (6.8) that

$$\int_0^{\infty} d\tau\, \rho(\tau) \frac{\tau - \tau(z)}{1 + z\Psi(z;\varepsilon)[\tau - \tau(z)]} = 0 \tag{6.10}$$

where the average is taken over the waiting time distribution $\rho(\tau)$. The function $\Psi(z;\varepsilon)$ can be obtained from (6.7) by Fourier inversion

$$\Psi(z;\varepsilon) \equiv \tilde{g}_{\mathbf{00}}(z;\varepsilon) = (2\pi)^{-d} \int_{1\mathrm{BZ}} d\mathbf{q}[z\tau(z) + \Phi(\mathbf{q})]^{-1} \tag{6.11}$$

where comparison of (6.2) and (6.11) shows that the small-z and ε behavior of $\Psi(z;\varepsilon)$ is given by (6.5) with $\langle \tau \rangle$ replaced by $\tau(z)$.

In summary, using the EMA we have determined the dominant small-z singularity in $G(\mathbf{q}, z; \varepsilon) \simeq g(\mathbf{q}, z; \varepsilon)$ in (6.7). On account of (3.10) and (4.12) this yields for the spectral densities $S_I(\omega)$ and $S_R(\omega)$ at low frequencies

$$S_I(\omega) \sim \Re e\, g(\mathbf{0}, i\omega; \varepsilon) \sim \Re e[i\omega\tau(i\omega)]^{-1} \tag{6.12}$$

The frequency-dependent waiting time $\tau(z)$ of the effective medium can be determined by solving (6.10), i.e., $\langle t_{\mathbf{00}} \rangle = 0$ for a given distribution $\rho(\tau)$. The results are valid for strong and weak disorder. In the former case we have to calculate $\tau(z)$ for a given $\rho(\tau)$ and compare (6.12) with $1/\omega$. This will be done in Section 7. In the latter case they reduce to the results (6.4) of the fluctuation expansion.

The EMA results are also very interesting because they allow us to discuss the crossover from weak to strong disorder. We find that the weak disorder results (6.6) for the d-dimensional SRJM are valid under the condition that $\langle \tau^{1+d/2} \rangle$ is finite. For instance, the well-known "long time tail" (6.6) in the Burnett function of a three-dimensional SRJM, $(\langle \tau^2 \rangle - \langle \tau \rangle^2)\sqrt{\omega}$, is *not valid* when $\langle \tau^2 \rangle < \infty$, but $\langle \tau^{5/2} \rangle$ is divergent. These points will be discussed in the next section by studying some examples.

7. SPECTRAL DENSITIES FOR STRONG DISORDER

In this section we construct three examples of strongly disordered systems, which successively better model the observed noise spectra, and discuss the validity of the EMA results.

If the waiting time distribution $\rho(\tau)$ vanishes or decays exponentially at large τ, all moments $\langle \tau^n \rangle$ $(n=0, 1, 2,...)$ exist and the system is weakly disordered for sure. To have a strongly disordered system we need a distribution with an algebraic tail $\rho(\tau) \sim \tau^{-2-a}$ where $a>0$, such that $\langle \tau \rangle$ is finite. As a *first example* we consider the (normalized) distribution

$$\rho(\tau) = (1+a)\, \tau_0^{1+a} \tau^{-2-a} \theta(\tau - \tau_0) \tag{7.1}$$

where $\theta(x)$ is the unit step function. The average waiting time for this distribution is $\langle \tau \rangle \equiv 1/\nu = \tau_0(1+1/a)$. For $0<a<1$ the second moment is divergent; for $1<a<2$ the moment $\langle \tau^3 \rangle$ is divergent, etc. From the EMA condition (6.12) we find for small z

$$\begin{aligned} \tau(z) = \langle \tau \rangle - z\Psi \langle [\tau - \tau(z)]^2 + \cdots \rangle \\ + (-z\Psi)^n \langle [\tau - \tau(z)]^{n+1} / \{1 + z\Psi[\tau - \tau(z)]\} \rangle \end{aligned} \tag{7.2}$$

where $a<n<a+1$, such that $\langle \tau^n \rangle$ is finite, and $\langle \tau^{n+1} \rangle$ is divergent. The last term gives a singular contribution $\sim (z\Psi)^a$:

$$\tau(z) = \langle \tau \rangle - \kappa_2 z \Psi + \cdots - (\pi a \langle \tau \rangle / \sin \pi a)(z \tau_0 \Psi)^a + \cdots \tag{7.3}$$

To determine the leading small-z singularity in $\tau(z)$ we also need the small-z and small-field behavior of $\Psi(z; \varepsilon)$, given by (6.5) with $\langle \tau \rangle$ replaced with $\tau(z)$. It reads

$$\Psi(z; \varepsilon) \simeq \Psi(0, 0) + \cdots + A[z\tau(z) + \varepsilon^2]^{d/2-1} \tag{7.4}$$

for $d \geqslant 2$, where the second term is multiplied by $(-)\log[z\tau(z) + \varepsilon^2]$ for $d=$even. In the drift-dominated regime $(z \ll \nu\varepsilon^2)$ the function $\Psi(z; \varepsilon)$ is regular in z and the leading singularity in $\tau(z)$ is always z^a. Thus in this frequency regime the results for weak disorder are only valid if *all* $\langle \tau^n \rangle$ $(n=1, 2, 3,...)$ *are finite.* In the diffusion-dominated regime $(\nu\varepsilon^2 \ll z \ll \nu)$ or in the *field free case* $(\varepsilon=0,\ z \ll \nu)$ the dominant singularity in $\Psi(z; \varepsilon)$ is $z^{d/2-1}$, and therefore the dominant singularity in (7.3) is

$$\tau(z) = \langle \tau \rangle \begin{cases} 1 + A_1 z^{1/2} + B_1 z^{a/2} + \cdots & (d=1) \\ 1 + A_2 (z \log z) + B_2 (z \log z)^a + \cdots & (d=2) \\ 1 + A_d z^{d/2} + B_d z^a & (d>2) \end{cases} \tag{7.5}$$

All coefficients A_d, B_d $(d=1, 2,...)$ can be calculated if so desired.

In conclusion: If the terms with A coefficients (or B coefficients) dominate we have weak (or strong) disorder. The weak-disorder results represent the dominant small-frequency singularity if $a > 1$ (that is, if $\langle \tau^2 \rangle < \infty$) for dimensionality $d \leqslant 2$; if $a > d/2$ (that is, if $\langle \tau^{1+d/2} \rangle < \infty$) they dominate for dimensionality $d > 2$.

Next, we use (7.3) for an explicit calculation of the spectral densities (6.12) in the case of strong disorder. The result for the excess current noise in the drift-dominated regime ($\omega \ll v\varepsilon^2$) is

$$S_I^{\text{exc}}(\omega) = \frac{I^2}{Nf} \frac{a}{\cos(\pi a/2)} \begin{cases} (\tau_0 \omega/2\,|\varepsilon|)^a & (d=1) \\ [(\tau_0 \omega/2\pi)\,|\log \varepsilon|\,]^a & (d=2) \\ [\tau_0 \omega \Psi(0;0)]^a & (d \geqslant 3) \end{cases} \tag{7.6}$$

and in the diffusion-dominated regime ($v\varepsilon^2 \ll z \ll v$):

$$S_I^{\text{exc}}(\omega) = \frac{I^2}{Nf} \frac{a}{\cos(\pi a/2)} \begin{cases} (2 \cos \tfrac{1}{4}\pi a)^{-1} (\tau_0^2 \omega v/4)^{a/2} & (d=1) \\ [(\tau_0 \omega/4\pi)\,|\log \tau_0 \omega|\,]^a & (d=2) \\ [\tau_0 \omega \Psi(0;0)]^a & (d \geqslant 3) \end{cases} \tag{7.7}$$

The resistance noise $S_R(\omega)$ in absence of a field is also determined by (7.7). Notice that $S_I^{\text{exc}}(\omega)$ for $d > 2$ is the same in both low-frequency regimes. The result (7.7) for $d = 1$ was reported before Ref. 26, where it was obtained by exactly solving Dyson–Schmidt-type integral equations. Note that the current noise for $d = 1$ in (7.7) is proportional to I^2, but in (7.6) to I^{2-a}, since I is proportional to ε for small ε. Related non-Ohmic behavior of the current itself in random barrier models for $d \leqslant 2$ was discussed by Movaghar *et al.*[27]

Even more interesting is the frequency dependence in (7.6) and (7.7). In all cases one sees that lowering the exponent a (increasing the disorder) makes the low-frequency divergence stronger. In the limit $a \downarrow 0$ the excess spectral density is seen to be inversely proportional to the frequency, i.e., it shows $1/f$ noise. However, (7.7) is not valid as $a \downarrow 0$ at *fixed frequency*, since $\omega < v \simeq a/\tau_0$ in (7.7).

Thus, the strongly disordered system with $\rho(\tau) \sim \tau^{-2-a}$ ($a > 0$) gives a spectral density of excess current and resistance noise, $S(\omega) \sim \omega^{-\gamma}$ with $\gamma = 1 - a$, where a may be chosen arbitrarily small. This functional form holds to arbitrarily small ω, and the integrated total intensity (3.12) is finite and proportional to $(v_\infty/v - 1) = [a(a+2)]^{-1}$.

However, the actual range of γ values, found in experiments, is $0.9 \lesssim \gamma \lesssim 1.3$. For $\rho(\tau)$ with an algebraic tail described by (7.1) the exponent γ remains always below unity. We therefore present as a second example a distribution which behaves as $S(\omega) \sim 1/\omega$ in a large range of frequencies. Consider the distribution

$$\rho(\tau) = A\tau^{-2}\theta(\tau - \tau_0)\,\theta(\tau_+ - \tau) \tag{7.8}$$

with $1/A = 1/\tau_0 - 1/\tau_+$, where $\tau_0/\tau_+ \ll 1$ (e.g., if $1/f$ noise is to be found in a frequency range of five decades, typically $\tau_0/\tau_+ < 10^{-7}$). The upper cutoff τ_+ is needed to keep the average waiting time $v^{-1} = \langle \tau \rangle \simeq \tau_0 \log(\tau_+/\tau_0)$ finite. In the frequency domain $1/\tau_+ \ll z \ll 1/\tau_0$ we now find from the EMA condition (6.10)

$$\tau(z) \simeq \tau_0 \log[\tau_0 z \Psi(z; \varepsilon)]^{-1} \tag{7.9}$$

In three dimensions we obtain for $z \ll v$ from (3.10), (6.12), and (7.9)

$$S_I^{\text{exc}}(\omega) = (I^2/Nf) \log(\tau_+/\tau_0)\{[\log \tau_0 \omega \Psi(0; 0)]^2 + \pi^2/4\}^{-1} \tag{7.10}$$

This result is valid for $1/\tau_+ \ll \omega \ll 1/\tau_0$ and is close to $1/f$ noise. For very small frequencies ($|z| = \omega \ll 1/\tau_+$ the EMA waiting time $\tau(z)$ is regular in $z = 0$, and the current noise becomes white, $S(\omega) \sim$ const.

However, if we compare (7.10) with (1.1) we find that the terms in (7.10) which play the role of the Hooge factor α_H in (1.1) (which is typically 10^{-3}) are not small, but of order unity. This is related to the fact that we have allowed each site to be a deep well. It seems more relevant to consider situations where only a small fraction p $(0 < p \ll 1)$ of the sites is a deep well (diluted randomness), the other fraction $1 - p$ being "host" sites of a regular uniform lattice with a fixed waiting time τ_0. As can be seen from (7.10), the diluted randomness is not a mechanism for explaining $1/f$ noise, but only a refinement of the model to account for a small Hooge factor.

We also allow for another refinement to account of a noise exponent γ in $S(\omega) \sim \omega^{-\gamma}$, different from unity. As can be seen in the first example, such behavior can be modeled by a power law distribution $\rho(\tau)$ as in (7.1).

Taking into account this aspect and the dilution we consider as a third example the following distribution of waiting times:

$$\rho(\tau) = (1 - p)\,\delta(\tau - \tau_0) + p(1 + a)\,\tau_0^{1+a}\tau^{-2-a}\theta(\tau - \tau_0)\,\theta(\tau_+ - \tau) \tag{7.11}$$

The exponent a may take values $-1 < a < 1$. The cutoff τ_+ (again with $\tau_+ \gg \tau_0$) is needed for $a < 0$ in order to keep $\langle \tau \rangle$ and thus the resistance finite:

$$\frac{1}{v} = \langle \tau \rangle = \tau_0 \left\{1 - p + p\frac{1 + a}{a}\left[1 - \left(\frac{\tau_0}{\tau_+}\right)^a\right]\right\} \tag{7.12}$$

The EMA condition (6.10) yields in the intermediate-frequency range, where $1/\tau_+ \ll z\Psi \ll 1/\tau_0$,

$$\tau(z) \simeq \tau_0\{1 + p/a - [p\pi(1+a)/\sin \pi a][z\tau_0 \Psi(z;\varepsilon)]^a\} \tag{7.13}$$

and again $\tau(z)$ is regular for extremely low frequencies $\tau_+ z\Psi \ll 1$. The spectral densities of excess current and resistance fluctuations follow from (6.12). In this example we have several small parameters: τ_0/τ_+, ε, and p, which forces us to divide the low-frequency regime into several subregimes. The case $\tau_0/\tau_+ \ll \varepsilon^2 < 1$ is of relevance only here.

For extremely low frequencies ($\omega \ll 1/\tau_+$) the noise is white; in the drift-dominated regime ($1/\tau_+ \ll \omega \ll \varepsilon^2/\tau_0$) the noise has the same frequency dependence as in (7.6) with a somewhat different coefficient that can be easily calculated. In the diffusion-dominated low-frequency regime ($\varepsilon^2 \ll \omega\tau_0 \ll 1$) the singular term in (7.13) would for $a<0$ dominate the constant term if $z\tau_0 \lesssim p^{1/|a|}$. However, for typical p values ($p \lesssim 0.1$) and typical (negative) a values ($|a| \lesssim 0.1$) this is never the case. Thus in the case of diluted randomness one has for all a the inequality $p(z\tau_0)^a \ll 1$. Hence the excess current noise, obtained from (6.12) and (7.13), has the form

$$S_I^{\text{exc}}(\omega) \sim p(I^2/N)(\omega/\omega_0)^{-\gamma} \tag{7.14}$$

with

$$\gamma = \begin{cases} 1 - a/2 & (d=1) \\ 1 - a & (d \geqslant 2) \end{cases}$$

where all relevant prefactors have been absorbed in the definition of ω_0. Thus the noise level or Hooge factor $\alpha_H \sim 10^{-4}$–10^{-3} in (1.1) is essentially the (small) concentration p of deep wells. It shows that each deep well independently gives a contribution to the noise spectrum. Equation (7.14) is universal in the sense that the cutoff times τ_0 and τ_+, introduced in (7.11), do not enter (7.14) in the case of true $1/f$ noise ($\gamma = 1$).

Before concluding this section it is of interest to use example (7.11) also to calculate the current noise for "hot electrons," where $\varepsilon = elE/2kT$ is large compared to unity. The EMA result (7.13) is still valid for small z, and $\Psi(z;\varepsilon) \simeq e^{-\varepsilon}$ for $\varepsilon \gg 1$ on account of (6.11). To simplify the calculations we put $a=0$ in (7.13), so that

$$\tau(z) \simeq \tau_0[1 - p - p\log(z\tau_0 e^{-\varepsilon})] \tag{7.15}$$

where $|p \log z\tau_0| \ll 1$. The current noise is then

$$S_I^{\text{exc}}(\omega) \simeq \alpha_H(E)(I^2/Nf) \tag{7.16}$$

with a field-dependent Hooge factor

$$\alpha_H(E) \simeq p/(1+p\varepsilon)^2 = \alpha_H/(1+E/E_c)^2 \tag{7.17}$$

where the critical field is $E_c = 2kT/(pel)$.

For a typical situation $p \simeq 10^{-3}$ and a lattice distance (mean free path) $l \simeq 10^4$ Å one has $E_c \simeq 10^7$ V/m which is a physically unrealistic value. A similar reduction of the noise level for hot electrons has been observed experimentally by Kleinpenning[28] and Bosman *et al.*[29] For the case $a \neq 0$ in (7.13) we obtain essentially the same results (7.16) and (7.17) if $\varepsilon\,|a| < 1$, whereas $\alpha_H(E)$ decays exponentially $\sim \exp(-|a|\,\varepsilon)$ for $\varepsilon\,|a| \gtrsim 1$.

In the last part of this section we investigate the validity of the EMA in the case of strong disorder and calculate the first nonvanishing correction to $\langle \tilde{G}_{\mathbf{kn}} \rangle \simeq \tilde{g}_{\mathbf{kn}}$ in (6.9). According to (3.10) and (4.11) we need the Fourier transform (2.10) of Eq. (6.9) at $\mathbf{q} = \mathbf{0}$:

$$G(\mathbf{0}, z; \varepsilon) = \frac{1}{z\tau(z)} + \frac{z^4}{[z\tau(z)]^2} \frac{1}{N} \sum_{\mathbf{m} \neq \mathbf{n}} \tilde{g}_{\mathbf{mn}} \tilde{g}_{\mathbf{nm}} \tilde{g}_{\mathbf{mn}} \langle t^2 \rangle^2 + \cdots \tag{7.18}$$

The double sum yields for small z and $d < 3$ in the diffusion-dominated regime ($\varepsilon^2 v \ll z \ll 1$):

$$\frac{1}{N} \sum_{\mathbf{m} \neq \mathbf{n}} \tilde{g}_{\mathbf{mn}} \tilde{g}_{\mathbf{nm}} \tilde{g}_{\mathbf{mn}} = (2\pi)^{-2d} \int d\mathbf{q} \int d\mathbf{q}'\, g(\mathbf{q})\, g(\mathbf{q}')\, g(\mathbf{q}+\mathbf{q}')$$
$$- [\Psi(z)]^3 \sim [z\tau(z)]^{d-3} \tag{7.19}$$

The integrals are estimated by power counting and Ψ^3 may be neglected on account of (6.5). In the drift-dominated regime ($z \ll v\varepsilon^2$) the right-hand side of (7.19) approaches a constant value as $z \to 0$. For the calculation of $\langle t^2 \rangle$ in (7.18) we restrict our discussion to the third example (7.11) with $p = 1$, i.e., $\rho(\tau) \sim \tau^{-2-a}$ and $a < d/2$, and we find from (6.8) and (6.10):

$$\langle t^2 \rangle = \int d\tau\, \rho(\tau) \frac{[\tau - \tau(z)]^2}{\{1 + z\Psi[\tau - \tau(z)]\}^2} \sim (z\Psi)^{a-1} \tag{7.20}$$

and (7.18) becomes

$$G(\mathbf{0}, z; \varepsilon) \sim (z\tau)^{-1} [1 + z^d \tau^{d-4} (z\Psi)^{2a-2} + \cdots] \tag{7.21}$$

where $\tau = \tau(z)$ has the value $\tau(z) \sim 1 + (z\Psi)^a$ because of (6.3) with $a < d/2$. For dimensionality $2 < d \leqslant 3$, where $\Psi(0)$ is finite, we find

$$G(\mathbf{0}, z; \varepsilon) \sim \begin{cases} z^{-1}(1 + z^a + z^{d-2+2a} + \cdots) & (a > 0) \\ z^{-1-a}(1 + z^{(d-2)(a+1)} & (a < 0) \end{cases} \tag{7.22}$$

Thus for all $a > -1$ the correction terms to the EMA are always negligible both in diffusion and in the drift-dominated regime for $2 < d \leqslant 3$.

The same conclusions apply to the drift-dominated regime for $d \leqslant 2$. In the diffusion-dominated regime we find similarly for $d = 2$ and $a \geqslant 0$

$$G(\mathbf{0}, z; \varepsilon = 0) \simeq (z \log z)^{-1}[1 + z^{2a}(\log z)^{2a-4} + \cdots] \qquad (a \geqslant 0) \qquad (7.23)$$

For $a < 0$ the higher correction terms in (7.18) are always dominating the EMA result. It also follows from (7.23) for $a = 0$ that the *first non-EMA-contribution* to (7.10) is of relative order $(\log \omega)^{-4}$. This contribution is of the same order of magnitude as the EMA contribution for the relevant frequencies.

Next, we consider the diffusion-dominated regime for $d < 2$, where $\Psi(z) \sim [z\tau(z)]^{(d-2)/2}$. For $a > 0$ and $0 < d < 2$ we have $\tau(z) \sim 1 + z^{ad/2}$ from (6.3) and (7.21) gives

$$G(0, z; \varepsilon) \sim z^{-1}(1 + z^{ad/2} + z^{ad} + \cdots) \qquad (7.24)$$

The term $\sim z^{ad/2}$ is an EMA contribution; the term $\sim z^{ad}$ is not from EMA. For $a \downarrow 0$ (which is the case of main interest for $1/f$ noise) the EMA breaks down, because both terms are of comparable size. For an exact approach in the one-dimensional case, see Ref. 26.

For $a < 0$ and $0 < d < 2$ we find from (6.3) the estimate $\tau(z) \sim z^{\beta(a)}$ with $\beta(a) = ad/(2 + 2a - da)$. Here (7.21) gives

$$G(\mathbf{0}, z; \varepsilon) \sim z^{-1-\beta(a)}[1 + O(z^0)] \qquad (7.25)$$

and the EMA- and non-EMA contributions are of equal size.

In conclusions, for dimensionality $d \leqslant 2$ and the parameter $a \lesssim 0$ the EMA is incorrect in the diffusion dominated regime. In the drift dominated regime the EMA is correct, but the current noise is not proportional to I^2 for small currents; cf. (7.6). However, in the context of $1/f$ noise only the case of diluted randomness with $p \ll 1$ [see Eq. (7.11] is physically relevant. Here one can easily verify, using similar arguments as above, that the non-EMA contributions are $O(p^2)$ and the EMA contributions $O(p)$. Thus for diluted randomness the EMA correctly describes the low-frequency behavior of $G(\mathbf{0}, z; \varepsilon)$ for *all* values of d, even in the case of strong disorder.

8. CONCLUSION

We have considered a hopping model, called symmetric random jump rate model (SRJM), on a d-dimensional lattice with quenched random jump rates $w_{\mathbf{n}} \sim 1/\tau_{\mathbf{n}}$, associated with each site $\mathbf{n}$. This model admits mobility fluctuations, but no fluctuations in the number of charge carriers.

Our main purpose was to relate the low-frequency behavior of excess current noise $S_I^{\text{exc}}(\omega)$ and resistance noise $S_D(\omega)$ to the "long time tails" occurring in diffusional systems with static disorder. For weak disorder we find $S(\omega) \sim [\langle \tau^2 \rangle - \langle \tau \rangle^2]\, \omega^{(d-2)/2}$, multiplied with a factor $\log(1/\omega)$ for d = even. For strong disorder, described by a distribution of waiting times $\rho(\tau) \sim \tau^{-2-a}$ with $-1 < a < 1$, $S(\omega) \sim \omega^{-1+a/2}$ for $d=1$ and $S(\omega) \sim \omega^{-1+a}$ for $d \geq 2$. The noise level or Hooge factor α_H, defined in (1.1), equals the fraction of sites with "defects" (deep wells). The low-frequency behavior is only valid for $\omega \gg 1/\tau_+$, which is the inverse of the large cutoff time in $\rho(\tau)$, not determined within the model.

Thus, only for strongly disordered systems, where the variance $\langle (\tau - \langle \tau \rangle)^2 \rangle$ is divergent (for an exception see discussion at the end of Section 6), the spectral densities $S_I^{\text{exc}}(\omega)$ and $S_D(\omega)$ with $a \approx 0$ resemble $1/f$ noise.

The integrated noise is expressed in terms of the dc resistance and the high-frequency resistance, which reaches a plateau value; cf. (3.12) and (4.20).

The SRJM differs in one respect to markedly from the standard models for $1/f$ noise.[1] In the standard models the average current correlation in the steady state decays as $\exp(-t/\tau)$ and the corresponding spectrum behaves as $S(\omega) \sim \tau/(1+\omega^2\tau^2)$. If one assumes a distribution $\rho(\tau)$ of characteristic waiting times behaving approximately as $\rho(\tau) \sim \tau^{-1}$ in a large interval $\tau_0 < \tau < \tau_+$, then the average spectrum shows $1/f$ behavior for $1/\tau_+ \ll \omega \ll 1/\tau_0$, where τ_0 and τ_+ are defined below (7.8). This type of waiting time distribution has been experimentally observed in relaxation processes, which are thermally activated,[1] or controlled by tunneling processes.[12,30] In the former case $\tau_{\mathbf{n}} \simeq \tau_0 \exp(U_{\mathbf{n}}/kT)$, where $U_{\mathbf{n}}$ is a random activation energy; in the latter case $\tau_{\mathbf{n}} \simeq \tau_0 \exp(a\lambda_{\mathbf{n}})$, where $\lambda_{\mathbf{n}}$ is a random tunneling distance. Both variables have a broad distribution $\mathfrak{D}(U) \sim$ const and $\mathfrak{D}(\lambda) \sim$ const, if $\rho(\tau) \sim 1/\tau$. However, there are no microscopic theories to explain a broad distribution $D(U)$.

The difference with the standard models shows clearly in the average current correlation $\langle \tau \exp(-t/\tau) \rangle$ and in the average spectral density $S(\omega) \sim \langle \tau^2/(1+\omega^2\tau^2) \rangle$. The additional factor τ inside the average [cf. T in (2.9)] comes from the steady state distribution (2.3) where a carrier has a probability $\sim \tau$ of being at a site with waiting time τ. Thus, the SRJM shows $1/f$ behavior in $S(\omega)$ if one assumes a waiting time distribution behaving approximately as $\rho(\tau) \sim 1/\tau^2$ or equivalently a *broad distribution*, $\tilde{\rho}(w) \sim$ const, of *transition rates* $w_{\mathbf{n}} \sim 1/\tau_{\mathbf{n}}$. If the transition rates $w_{\mathbf{n}}$ are related to a thermally activated process with activation energy $U_{\mathbf{n}}$, i.e.,

$w_n = w_0 \exp(-U_n/kT)$, where w_0 is a characteristic attempt frequency, we find a *thermal* distribution of the activation energies: $D(U) \sim \exp(-U/kT)$. In the standard theory one has a *broad* distribution $D(U)$. Also in the present case we are unable to derive this distribution from a microscopic model.

Therefore, both the SRJM and the standard hopping models with quenched disorder reduce the explanation of the dynamic phenomenon of $1/f$ noise to the explanation of the static disorder. Each model has its own advantages and disadvantages. As an example of a standard model Machta *et al.*[11] have considered a two-state hopping model. This model has the attractive feature of a broad distribution of $\log \tau$, as found in experiments.[1] However, it shows no mobility fluctuations, but only fluctuations in the number of charge carriers. According to Hooge *et al.*[2] experiments confirm that mobility fluctuations and not number fluctuations are in general the dominant noise source in resistance fluctuations. In the SRJM the situation is reversed: $\log \tau$ does not have a broad distribution, but an exponential one; it shows mobility fluctuations but no number fluctuations.

Furthermore, we have given a microscopic derivation of the equality of excess current and resistance noise, and related these quantities to the four-point-velocity correlation and the Burnett correlation function. These equalities are valid for weak and strong disorder.

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APPENDIX

A standard procedure[21] in generalized hydrodynamic (here the electric field is vanishing) is to express the response function (2.7)

$$\mathfrak{F}(\mathbf{q}, z) = \{z + q_x^2 \hat{U}(\mathbf{q}, z)\}^{-1} \tag{A1}$$

in terms of a generalized transport coefficient $\hat{U}(\mathbf{q}, z)$. In order to simplify the discussion we have taken $\mathbf{q}$ along the x axis. $\hat{U}$ has a $\mathbf{q}$ expansion:

$$q_x^2 \hat{U}(\mathbf{q}, z) = q_x^2 \hat{U}_2(z) - q_x^4 \hat{U}_4(z) + \cdots \tag{A2}$$

where [cf. (3.4) and (3.5)]

$$\hat{U}_n(z) = -\frac{1}{n!}\left(i\frac{\partial}{\partial q_x}\right)^n [\mathfrak{F}(\mathbf{q}, z)]^{-1}\Big|_{\mathbf{q}=\mathbf{0}} \tag{A3}$$

Since q_x derivatives of the response function yield moments of the displacements it is straightforward to verify that the inverse Laplace transforms $U_n(t)$ satisfy

$$\begin{aligned} U_2(t) &= \langle v_x(t)\, v_x(0)\rangle \\ U_4(t) &= \int_0^t d\tau \int_0^\tau d\tau' [\langle v_x(t)\, v_x(\tau)\, v_x(\tau')\, v_x(0)\rangle \\ &\quad - \langle v_x(t)\, v_x(\tau)\rangle \langle v_x(\tau')\, v_x(0)\rangle] \end{aligned} \tag{A4}$$

where $U_2(t)$ equals the VACF (3.7) and $U_4(t)$ is the modified Burnett function. The usual Burnett correlation function $\beta(t)$ defined in (5.1), differs from this expression because the cumulant $\langle\langle \cdots \rangle\rangle$ contains two additional terms with a product of two VACF's [see (1.14)].

For the present model these additional terms are $D^2[\delta(t-\tau')\,\delta(\tau) + \delta(t)\,\delta(\tau-\tau')]$ according to (5.5), and do not contribute to (A4). Thus we have for the SRJM

$$U_4(t) = \beta(t) \tag{A5}$$

which is a special property of this model, and not valid for general models (see, e.g., Ref. 13). Combination of (A5) and (A3) also yields (5.2).

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